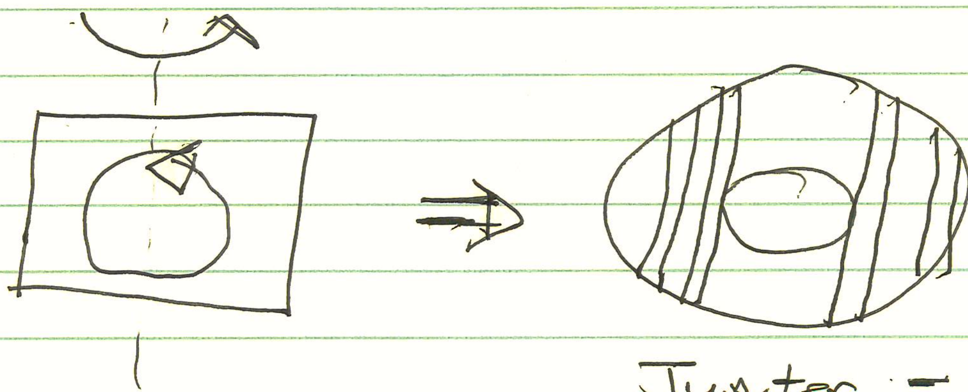
Lecture VI

Rotating Convection

Now,



Jupiter – bands

– $T_{day} \sim 10$ hrs.

how do convection and rotation interact?

Key point: rotation changes the freezing-in law for vorticity.

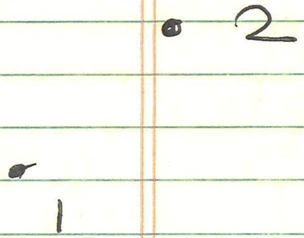
Recall: $$\boxed{\partial_t \underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\nabla} \times (\underline{v} \times \underline{\omega})}$$

induction equation with vorticity.

$$\underline{\nabla} \cdot \underline{v} = 0 \Rightarrow \begin{cases} \partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla}\, \underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{V} \\ \dfrac{d\underline{\omega}}{dt} - \nu \nabla^2 \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{V} \end{cases}$$

Why "freezing-in"?

Consider two test particles at $\underline{r}_1$, $\underline{r}_2$, "frozen into" flow, i.e.



$$\frac{d\underline{u}}{dt} = -\gamma\left(\underline{u} - V(\underline{x},t)\right) + \underline{\tilde{f}}$$

$\gamma \to \infty$ (freezing in):

$$\frac{d\underline{r}_1}{dt} = \underline{V}(\underline{r}_1, t)$$

$$\frac{d\underline{r}_2}{dt} = \underline{V}(\underline{r}_2, t)$$

let $\delta\underline{l} = \underline{r}_2 - \underline{r}_1$

think of as a filament

$$\frac{d\,\delta\underline{l}}{dt} = \underline{V}\left(\underline{r}_+ + \frac{\delta\underline{l}}{2}, t\right) - \underline{V}\left(\underline{r}_+ - \frac{\delta\underline{l}}{2}, t\right)$$

$$= \delta\underline{l} \cdot \underline{\nabla}\underline{V}$$

$\to$ $\delta\underline{l}$ follows the flow and is stretched by it.

Now,
$$\left.\begin{aligned} \frac{d\underline{\omega}}{dt} &= \underline{\omega} \cdot \underline{\nabla}\underline{V} \\ \frac{d}{dt}\delta\underline{l} &= \delta\underline{l} \cdot \underline{\nabla}\underline{V} \end{aligned}\right\}$$

$\underline{\omega}$, $\delta\underline{l}$ obey same equation

- so vorticity is frozen-into the flow (up to ν).

- vorticity filaments follow and are stretched by the flow.

- obviously, freezing-in related to Kelvin's Theorem.

Now, in rotating fluid:

$$\underline{\Omega} = \Omega \hat{z}$$

$$\underline{v}_{Lab} = \underline{v}_{Fluid} + \underline{\Omega} \times \underline{r}$$

$$\underline{v} \rightarrow \underline{v} + \underline{\Omega} \times \underline{r}$$

so, for $\left\{ \begin{array}{l} \text{isentropic} \\ \underline{\nabla} \cdot \underline{v} = 0 \end{array} \right.$, eqn. in frame:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\underline{\nabla} \left(\frac{p}{\rho} - \frac{\Omega^2 r^2}{2} \right) + \underline{v} \times 2\underline{\Omega}$$

centrifugal force (absorb into pressure)

Coriolis force

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = \underline{\nabla} \left(\frac{v^2}{2} \right) - \underline{v} \times \underline{\omega}$$

$\Rightarrow$

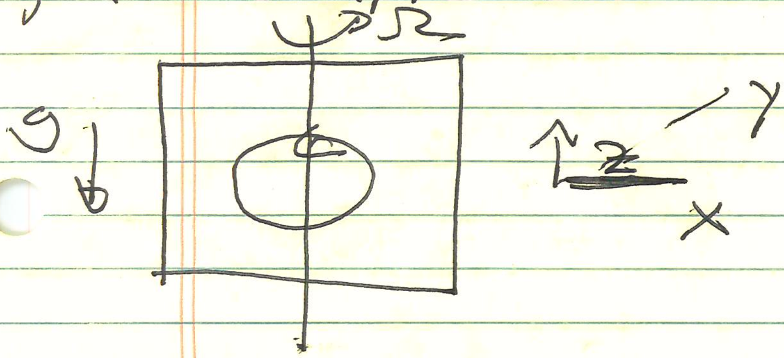
$$\frac{\partial \underline{v}}{\partial t} = -\underline{\nabla} \left(\frac{p}{\rho} + \frac{v^2}{2} - \Omega^2 r^2 \right) + \underline{v} \times \underline{\omega} + \underline{v} \times 2\underline{\Omega}$$

curl $\Rightarrow$

$$\frac{\partial}{\partial t}(\underline{\omega}+2\underline{\Omega}) = \underline{\nabla}\times\underline{v}\times(\underbrace{\underline{\omega}}_{\text{relative (fluid) vorticity}}+\underbrace{2\underline{\Omega}}_{\text{planetary vorticity}}) + \nu\nabla^2\underline{\omega}$$

- $\underline{\omega}+2\underline{\Omega}$ frozen-in
- $\int d\underline{a}\cdot(\underline{\omega}+2\underline{\Omega})$ conserved.

so, for simple case:



(tilts later)

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{v}\cdot\underline{\nabla}(\underline{\omega}+2\underline{\Omega}) - \nu\nabla^2\underline{\omega} = (2\underline{\Omega}+\underline{\omega})\cdot\underline{\nabla}\,\underline{v}$$

$$\frac{d\underline{\omega}}{dt} - \nu\nabla^2\underline{\omega} = (2\underline{\Omega}+\underline{\omega})\cdot\underline{\nabla}\underline{v}$$
$$= 2\Omega\,\partial_z\underline{v} + \underline{\omega}\cdot\underline{\nabla}\underline{v}$$

For: $\underbrace{\Omega}_{\text{mean vorticity}} \gg |\underline{\omega}|, |\nabla\underline{v}|$ etc

i.e. system rotation strong compared to relative vorticity.

- motions can't vary in direction of Ω
- all slow, steady motions of rotating inviscid fluid are necessarily 2D,

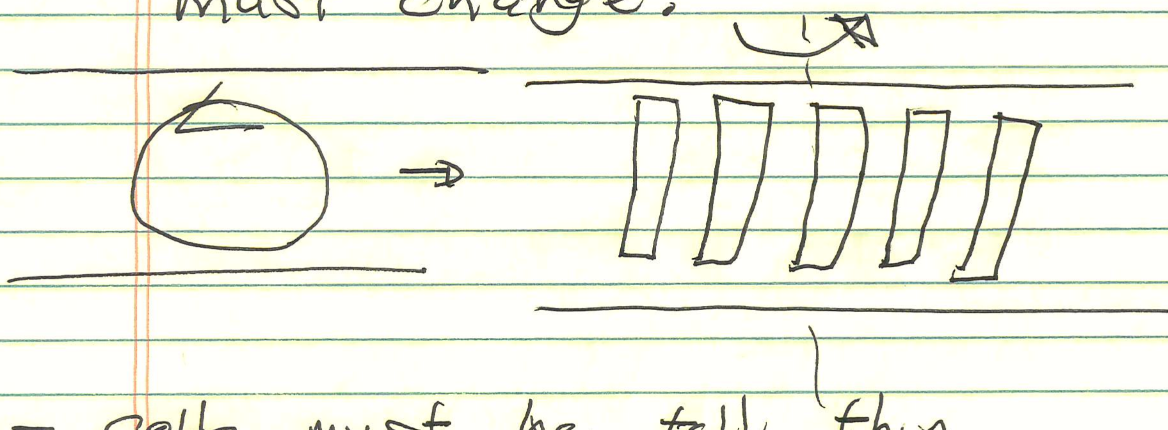
i.e.

$$\boxed{2\Omega\, \partial_z \underline{V} \cong 0}$$

→ Taylor - Proudman Theorem (1921, 1916)

→ Rapid rotation "two-dimensionalizes" the flow.

→ convection cells/modes organize into "Taylor Columns", "Proudman Pillars" → cartoon must change:



- cells must be tall, thin
- viscous BL at top, bottom (Ekman Layer)

i.e. $-\nu \partial_z^2 \underline{W} = 2\Omega\, \partial_z \underline{V} \neq 0$

$\ell \sim (\nu/\Omega)^{1/2}$ → thickness.

→ Physics of Taylor - Proudman Theorem ;

⟺ Vortex lines extract energy penalty when bent.

⟺ Vortex lines don't like being bent.

→ Inertial Waves → Radial Buoyancy waves in rotating fluid.

Recall HW Set 1:

$$\omega^2 = k_z^2 (4\Omega^2) / k_z^2 + k_r^2$$

and radial boundary condition sets eigenvalue.

Derivation:

From vorticity:

$$\frac{\partial \tilde{\underline{\omega}}}{\partial t} + \tilde{\underline{v}} \cdot \underline{\nabla} \left(\cancel{\tilde{\underline{\omega}}} + \cancel{2\underline{\Omega}} \right) = 2\Omega \, \partial_z \tilde{\underline{v}} + \cancel{\tilde{\underline{\omega}} \cdot \underline{\nabla} \tilde{\underline{v}}}$$

$$\partial_t \tilde{\omega}_z = 2\Omega \, \partial_z \tilde{v}_z$$

And from EOM ⇒

$\underline{\nabla} \times \underline{\nabla} \times \underline{v}$ ⇒

$$\partial_t \underline{\nabla}(\underline{\nabla}\cdot\tilde{\underline{V}}) - \partial_t \nabla^2 \tilde{\underline{V}}$$

$$= \underline{\nabla}\times\underline{\nabla}\times\left[\underline{V}\times 2\underline{\Omega}\,\hat{z}\right]$$

$$-\partial_t \nabla^2 \tilde{\underline{V}} = \underline{\nabla}\times(2\Omega\partial_z)\tilde{\underline{V}} - \cancel{\tilde{\underline{V}}\cdot\underline{\nabla}\,2\Omega\hat{z}}$$

so $\hat{z}$:

$$-\partial_t \nabla^2 \tilde{V}_z = 2\Omega\partial_z \tilde{\omega}_z$$

$$\partial_t \tilde{\omega}_z = 2\Omega\partial_z \tilde{V}_z$$

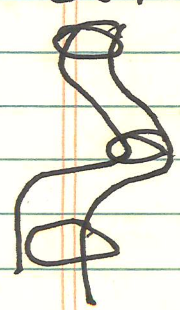
{axial gradient in $\tilde{\omega}$ ⇒ axial acceleration

{stretching vortex ⇒ vorticity fluctuation

$$\Rightarrow \omega^2 = k_z^2 4\Omega^2/(k_r^2+k_z^2)$$

- physics is rotating (vortex) flow lines dont like being 'bent' ($k_z \neq 0$) ⇒ imposes energy penalty for finite k_z motions.
  ⇒ ⊕ definite δW.

- rough picture is one of gyroscopic restoring force (conservation L_z)



- analogous Alfven wave in MHD.

- N.B. inertial wave
  - $v_{gr} < 0$
  - $\omega = 0$ finite k_r shear layers

The point:

→ Convection requires cellular overturning (i.e. $k_z > 1/ht_{box}$)
→ minimal k_z

→ $\gamma^2 \sim k_x^2 / k_z^2 + k_x^2$

⇒ low k_z motions favored.

⇒ [Rotation is (strong) stabilizing effect on convection, as energy coupled to inertial wave.

Another dimensionless number:

Ra, Pr

$Ta = 4\Omega^2 d^4/\nu^2$ — $d \sim$ box scale

$= 4\left(\Omega d^2/\nu\right)^2$

Taylor #

~ Taylor number captures natural competition between rotation and viscous diffusion.

~ Ta joins Ra, Pr as key parameters in convective stability theory

~ $Ra_{crit} = Ra_{crit}(Pr, Ta)$ is now stability threshold problem.

N.B. Ra, Ta both involve ν but are distinct – $\alpha g \Delta T/d$ vs Ω^2.

Can combine stationary convection and inertial wave calculations to obtain basic equations:

$$\partial_t \Theta = \beta W + \kappa \nabla^2 \Theta$$

$$\partial_t \nabla^2 W = g\alpha \nabla_h^2 \Theta + \nu \nabla^2 \nabla^2 W - 2\Omega\, \partial_z \omega_z$$

$$\partial_t \omega_z = -2\Omega\, \partial_z W$$

(derive)

notation as before

For ideal stability:

$$\boxed{\omega^2 = \left[-g\alpha\beta k_h^2 + (4\Omega_z^2) k_z^2\right] / k^2}$$

- favors high k_h, low k_z cells

$\Rightarrow$

- Taylor columns (thin) and Proudman pillars, as shown in movies.

- stabilizing effect of rotation evident $\Rightarrow$ cell anisotropy.

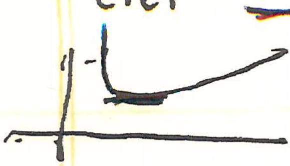
For viscous, conductive stability (rotational) with rotation, then:

$$Ra_{crit} = Ra_{crit}(Ta, \alpha), \quad \text{for } Pr \sim 1,$$

$$\downarrow$$

$$\underbrace{k_h h}$$

One can further specify Ra_α as Ra_{crit} minimum (vs $\alpha = k_h h$)



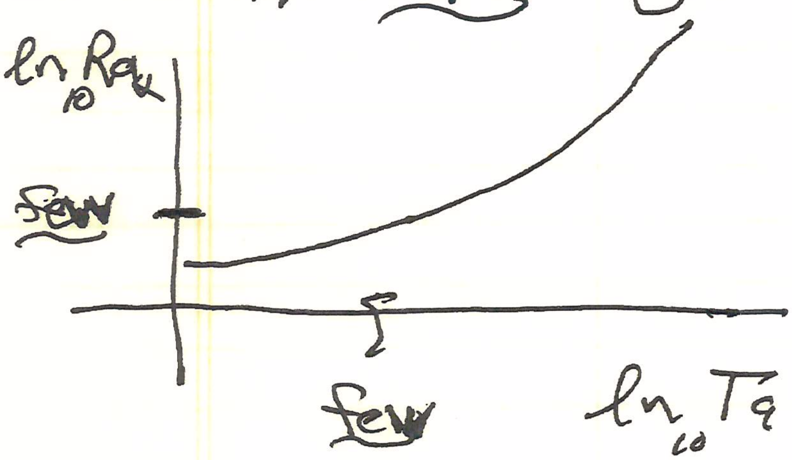
i.e. smallest $Ra_{crit}(Ta, \alpha)$ scanned over α.

$Ra_c = Ra_c(Ta)$

$\downarrow$

<u>Taylor # dependence</u>

For all cases: $\left\{\begin{array}{l}\text{no-slip}\\ \text{stress free}\end{array}\right.$



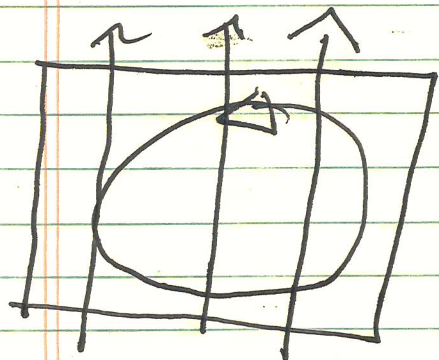
$\left\{\begin{array}{l}\text{demonstrates}\\ \text{stabilizing}\\ \text{effect rotation.}\end{array}\right.$

$Ta \uparrow \Rightarrow$
$Ra_{crit} \uparrow$

→ can develop variational principle for
aside exchange-of-stabilities case. Need
also treat over-stable limit.

→ Cultural Aside: <u>Magnetoconvection</u>

Dynamo-generated magnetic fields can feed back on convection. A particular example is sunspots (i.e. dark – lower T ?) – convection weakened), which are associated with strong $\underline{B}$ fields.



∇T

$\underline{B_0}$ $\downarrow \underline{g}$

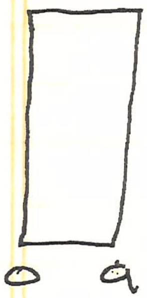
- similar to rotation problem with 'bending' due $\underline{B_0}$
- energy penalty.

- Alfven wave replaces inertial wave

Aside: what of tall, thin box.?

tall, thin box

Now:

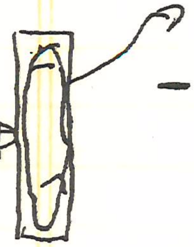


→ ignore top/bttm boundary

→ no-slip on side walls is key → and only relevant case

i.e. $V_z \Rightarrow w$; $w(0) = w(a) = 0$

side B.L.



side B.L. due no-slip

- long thin cell must fight no-slip b.c.'s on side wall

- higher k_h will introduce high $k_{\perp}$ damping.

∴ - expect high $Ra)_{crit}$ due side-wall no slip, even at $k_h \to k_{h\,min}$.

- curvature of Ra_{crit} vs k_h curve TBD. Speculate rather weak curvature.

- as side surface area >> top surface area, expect top no-slip vs others free comparison not significant.